q-ANALOG OF TABLEAU CONTAINMENT

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ABSTRACT. We prove a q-analog, which is valid if q is a positive real number, of the following: the probability that, when n tends to infinity, a randomly chosen standard Young tableau of size n contains a fixed standard Young tableau of shape $\lambda \vdash k$ is equal to $\frac{f^\lambda}{k!}$ (Journal of Combinatorial Theory Series A, volume 97, 117–128, 2002). We also consider pairs of tableaux.

1. INTRODUCTION

Let $\mathfrak{S}_n$ denote the set of permutations of $[n] = \{1, 2, \dots, n\}$. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $\mathfrak{S}_n$. Let $\pi_{\le k}$ (resp. $\pi_{>k}$) denote the permutation obtained from π by taking integers $i \le k$ (resp. $i > k$) and by order-preserving relabeling. Let $\pi^{\le k}$ (resp. $\pi^{>k}$) denote the permutation obtained from $\pi_1\pi_2\cdots\pi_k$ (resp. $\pi_{k+1}\pi_{k+2}\cdots\pi_n$) by order-preserving relabeling. For example, if $\pi = 513697428$, then $\pi_{\le 4} = 1342$, $\pi_{>4} = 12534$, $\pi^{\le 4} = 3124$ and $\pi^{>4} = 53214$. If $\sigma = \pi_{\le k}$ for some k, then we say that π *contains* σ.

Let $\mathfrak{I}_n$ denote the set of involutions in $\mathfrak{S}_n$. For a permutation $\sigma \in \mathfrak{S}_k$, let $\mathfrak{I}_n(\sigma)$ denote the set of involutions in $\mathfrak{I}_n$ containing σ, i.e.,

$$\mathfrak{I}_n(\sigma) = \{\pi \in \mathfrak{I}_n : \pi_{\le k} = \sigma\}.$$

Let n be a nonnegative integer. A *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of n, denoted by $\lambda \vdash n$, is a weakly decreasing sequence of positive integers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r$ summing to n. The *Ferres diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is the left-justified array of squares such that the i-th row has λ_i squares. We will identify a partition with its Ferres diagram. Let λ and μ be partitions such that the Ferres diagram of μ is contained in the Ferres diagram of λ. Then the *skew shape* λ/μ is the set-theoretic difference $\lambda \setminus \mu$, and we denote $\lambda/\mu \vdash n - m$ if $\lambda \vdash n$ and $\mu \vdash m$. A partition λ is also considered as the skew shape $\lambda/\emptyset$. A *standard Young tableau*, or a *SYT* for short, of shape $\lambda/\mu \vdash n$ is a filling of λ/μ with integers $1, 2, \dots, n$ such that entries are increasing along rows and columns. If T is a SYT of shape $\lambda/\mu \vdash n$, then we write $sh(T) = \lambda/\mu$ and we say that T is of size n.

Let T be a SYT of shape λ. Let $T_{\le k}$ (resp. $T_{>k}$) denote the SYT obtained from T by taking squares with integer $i \le k$ (resp. $i > k$) and by order-preserving relabeling. For example, if $T = \begin{array}{|c|c|c|c|}\hline 1&2&4&7\\ \hline 3&5&6\\ \cline{1-3} 8&9\\ \cline{1-2}\end{array}$, then $T_{\le 5} = \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array}$ and $T_{>5} = \begin{array}{cccc} &&&\boxed{2}\\ &&\boxed{1}&\\ \boxed{3}&\boxed{4}&&\end{array}$.
If $U = T_{\le k}$ for some k, then we say that T *contains* U.

Let A be a fixed SYT of shape $\alpha \vdash a$. Let $\mathcal{T}_n(A)$ denote the set of SYTs of size n containing A. We denote $\mathcal{T}_n = \mathcal{T}_n(\emptyset)$.

We will assume reader's familiarity with the Robinson-Schensted correspondence, for reference, see [6, 8]. If π corresponds to (P, Q) in the Robinson-Schensted correspondence, then we write $\pi \xleftrightarrow{\text{RS}} (P, Q)$, and also write $P(\pi) = P$ and $Q(\pi) = Q$. If π is an involution, then we write $\pi \xleftrightarrow{\text{RS}} P$.

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Let π be an involution and $\pi \xleftrightarrow{\mathrm{RS}} T$. We can easily see that T contains A if and only if π contains σ for some $\sigma \in \mathfrak{S}_a$ with $P(\sigma) = A$. Thus the Robinson-Schensted correspondence induces a bijection

$$RS : \mathcal{T}_n(A) \to \bigcup_{\sigma : P(\sigma)=A} \mathfrak{I}_n(\sigma). \tag{1}$$

McKay, Morse and Wilf [5] proved that if $\sigma \in \mathfrak{S}_a$, then

$$\lim_{n\to\infty} \frac{|\mathfrak{I}_n(\sigma)|}{|\mathfrak{I}_n|} = \frac{1}{a!}. \tag{2}$$

As a corollary, they obtained that the probability that, when n goes to infinity, a random SYT of size n contains A equals $\frac{f^\alpha}{a!}$, i.e.,

$$\lim_{n\to\infty} \frac{|\mathcal{T}_n(A)|}{|\mathcal{T}_n|} = \frac{f^\alpha}{a!}, \tag{3}$$

where f^α denotes the number of SYTs of shape α.

Jaggard [2] defined the 'j-set' $J(\pi)$ of a permutation π and found the following exact formula for $|\mathfrak{I}_{n+a}(\sigma)|$. For given $\sigma \in \mathfrak{S}_a$,

$$|\mathfrak{I}_{n+a}(\sigma)| = \sum_{\substack{j\in J(\sigma)\\ k=n-a+j}} \binom{n}{k} t_k, \tag{4}$$

where $t_k = |\mathfrak{I}_k|$, the number of involutions in $\mathfrak{S}_k$.

In this paper we find q-analogs of (2), (3) and (4).

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. A *descent* of π is an integer i such that $\pi_i > \pi_{i+1}$. Let T be a SYT. A *descent* of T is an integer i such that $i+1$ is in a row lower than the row containing i in T. Let $D(\pi)$ (resp. $D(T)$) denote the set of all descents of π (resp. T). Let $\mathrm{maj}(\pi)$ (resp. $\mathrm{maj}(T)$) denote the sum of all descents of π (resp. T).

Let $\pi \xleftrightarrow{\mathrm{RS}} (P,Q)$. It is well known that $D(\pi^{-1}) = D(P)$ and $D(\pi) = D(Q)$, see [8]. Thus $q^{\mathrm{maj}((\pi^{-1})^{>a})} = q^{\mathrm{maj}(P_{>a})}$ and $q^{\mathrm{maj}(\pi^{>b})} = q^{\mathrm{maj}(Q_{>b})}$ for any nonnegative integers a and b.

Let us define

$$\mathrm{imaj}(\pi) = \mathrm{maj}(\pi^{-1}), \quad A_n(p,q) = \sum_{\pi\in\mathfrak{S}_n} p^{\mathrm{imaj}(\pi)} q^{\mathrm{maj}(\pi)},$$

$$t_n(q) = \sum_{\pi\in\mathfrak{I}_n} q^{\mathrm{maj}(\pi)}, \quad f^{\lambda/\mu}(q) = \sum_{sh(T)=\lambda/\mu} q^{\mathrm{maj}(T)},$$

$$[n]_q! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}), \quad \begin{bmatrix} n\\ k\end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

Now we state our main results.

Theorem 1.1. *Let a be an integer and let $\sigma \in \mathfrak{S}_a$. Then*

$$\sum_{\pi\in\mathfrak{I}_{n+a}(\sigma)} q^{\mathrm{maj}(\pi^{>a})} = \sum_{\substack{j\in J(\sigma)\\ k=n-a+j}} q^{\mathrm{maj}(\sigma^{>j})} \begin{bmatrix} n\\ k\end{bmatrix}_q t_k(q).$$

For a real number $r > 0$, let us denote $\bar{r} = \min(r, r^{-1})$.

Theorem 1.2. *Let a be an integer and $\sigma \in \mathfrak{S}_a$. If $q > 0$, then*

$$\lim_{n\to\infty} \frac{\sum_{\pi\in\mathfrak{I}_n(\sigma)} q^{\mathrm{maj}(\pi^{>a})}}{\sum_{\pi\in\mathfrak{I}_n} q^{\mathrm{maj}(\pi^{>a})}} = \frac{q^{\mathrm{maj}(\sigma)} + (1-\bar{q})C}{[a]_q! + (1-\bar{q})D},$$

where C and D are polynomials of q and $\bar{q}$. (See Theorem 2.6 for exact value.)

Theorem 1.3. *Let A be a SYT of shape $\alpha \vdash a$. If $q > 0$, then*

$$\lim_{n\to\infty} \frac{\sum_{T\in\mathcal{T}_n(A)} q^{\mathrm{maj}(T_{>a})}}{\sum_{T\in\mathcal{T}_n} q^{\mathrm{maj}(T_{>a})}} = \frac{f^\alpha(q) + (1-\bar{q})E}{[a]_q! + (1-\bar{q})D},$$

where E and D are polynomials of q and $\bar{q}$. (See Theorem 3.3 for exact value.)

Similarly, we can consider pairs of SYTs. Let A and B be SYTs of shape $\alpha \vdash a$ and $\beta \vdash b$ respectively. For a pair (P,Q) of SYTs, we say that (P,Q) *contains* (A,B) if both P and Q contain A and B respectively. Let $\mathcal{T}_n(A,B)$ denote the set of pairs (P,Q) of SYTs of the *same shape* of size n containing (A,B). Let $\pi \xleftrightarrow{\mathrm{RS}} (P,Q)$. It is easy to see that (P,Q) contains (A,B) if and only if $\pi_{\le a} = \sigma$ and $\pi^{\le b} = \tau$ for some $\sigma \in \mathfrak{S}_a$ and $\tau \in \mathfrak{S}_b$ with $P(\sigma) = A$ and $Q(\tau) = B$. Let $\mathfrak{S}_n|_\sigma^\tau$ denote the set of $\pi \in \mathfrak{S}_n$ such that π contains σ and π^{-1} contains τ^{-1}, i.e.,

$$\mathfrak{S}_n|_\sigma^\tau = \{\pi \in \mathfrak{S}_n : \pi_{\le a} = \sigma, \pi^{\le b} = \tau\}.$$

Then the Robinson-Schensted correspondence induces the following bijection.

$$RS : \mathcal{T}_n(A,B) \to \bigcup_{\substack{\sigma : P(\sigma)=A \\ \tau : Q(\tau) = B}} \mathfrak{S}_n|_\sigma^\tau. \tag{5}$$

We define the 'j_2-set' $J(\sigma,\tau)$ of a pair (σ,τ) of permutations, and prove the following analogs of Theorem 1.1, 1.2 and 1.3.

Theorem 1.4. *Let a, b, n, m and ℓ be integers with $a + m = b + n = \ell$. Let $\sigma \in \mathfrak{S}_a, \tau \in \mathfrak{S}_b$. Then*

$$\sum_{\pi\in\mathfrak{S}_\ell|_\sigma^\tau} p^{\mathrm{imaj}(\pi_{>a})} q^{\mathrm{maj}(\pi^{>b})} = \sum_{\substack{j\in J(\sigma,\tau)\\ k=n-a+j}} p^{\mathrm{imaj}(\tau_{>j})} q^{\mathrm{maj}(\sigma^{>j})} \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q A_k(p,q).$$

Setting $p = q = 1$, we get the size of $\mathfrak{S}_\ell|_\sigma^\tau$.

Corollary 1.5. *Let a, b, n, m and ℓ be integers with $a + m = b + n = \ell$. Let $\sigma \in \mathfrak{S}_a, \tau \in \mathfrak{S}_b$. Then the number of elements of $\mathfrak{S}_\ell|_\sigma^\tau$ is equal to*

$$\sum_{\substack{j\in J(\sigma,\tau)\\ k=n-a+j}} \binom{m}{k}\binom{n}{k} k!.$$

Theorem 1.6. *Let a, b be integers and let $\sigma \in \mathfrak{S}_a$, $\tau \in \mathfrak{S}_b$. If $p, q > 0$, then*

$$\lim_{n\to\infty} \frac{\sum_{\pi\in\mathfrak{S}_n|_\sigma^\tau} p^{\mathrm{imaj}(\pi_{>a})} q^{\mathrm{maj}(\pi^{>b})}}{\sum_{\pi\in\mathfrak{S}_n} p^{\mathrm{imaj}(\pi_{>a})} q^{\mathrm{maj}(\pi^{>b})}} = \frac{p^{\mathrm{imaj}(\tau)} q^{\mathrm{maj}(\sigma)} + (1-\bar{p})(1-\bar{q})C'}{[b]_p!\,[a]_q! + (1-\bar{p})(1-\bar{q})D'},$$

where C' and D' are polynomials of $p, \bar{p}, q$ and $\bar{q}$. (See Theorem 2.7 for exact value). Specially, if $p = 1$ or $p = q = 1$, then we have

$$\lim_{n\to\infty} \frac{\sum_{\pi\in\mathfrak{S}_n|_\sigma^\tau} q^{\mathrm{maj}(\pi^{>b})}}{\sum_{\pi\in\mathfrak{S}_n} q^{\mathrm{maj}(\pi^{>b})}} = \frac{q^{\mathrm{maj}(\sigma)}}{b!\,[a]_q!},$$

$$\lim_{n\to\infty} \frac{|\mathfrak{S}_n|_\sigma^\tau|}{|\mathfrak{S}_n|} = \frac{1}{a!b!}.$$

Theorem 1.7. *Let A, B be SYTs of shape $\alpha \vdash a$, $\beta \vdash b$ respectively. If $p, q > 0$, then*

$$\lim_{n\to\infty} \frac{\sum_{(P,Q)\in\mathcal{T}_n(A,B)} p^{\mathrm{maj}(P_{>a})} q^{\mathrm{maj}(Q_{>b})}}{\sum_{(P,Q)\in\mathcal{T}_n(\emptyset,\emptyset)} p^{\mathrm{maj}(P_{>a})} q^{\mathrm{maj}(Q_{>b})}} = \frac{f^\beta(p) f^\alpha(q) + (1-\bar{p})(1-\bar{q})E'}{[b]_p!\,[a]_q! + (1-\bar{p})(1-\bar{q})D'},$$

where E' and D' are polynomials of $p, \bar{p}, q$ and $\bar{q}$. (See Theorem 3.4 for exact value). Specially, if $p = 1$ or $p = q = 1$, then we have

$$\lim_{n\to\infty} \frac{\sum_{(P,Q)\in\mathcal{T}_n(A,B)} q^{\mathrm{maj}(Q_{>b})}}{\sum_{(P,Q)\in\mathcal{T}_n(\emptyset,\emptyset)} q^{\mathrm{maj}(Q_{>b})}} = \frac{f^\beta f^\alpha(q)}{b!\,[a]_q!},$$

$$\lim_{n\to\infty} \frac{|\mathcal{T}_n(A,B)|}{|\mathcal{T}_n(\emptyset,\emptyset)|} = \frac{f^\alpha f^\beta}{a!b!}.$$

The rest of this paper is organized as follows. In Section 2, we define j_2-sets and prove Theorem 1.1, 1.2, 1.4 and 1.6. In Section 3, we prove Theorem 1.3 and 1.7. In Section 4, we find a simple method to determine a j_2-set.

2. Permutation containment

Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. Recall the definitions of $\pi^{\le k}$, $\pi^{>k}$, $\pi_{\le k}$ and $\pi_{>k}$. These are easy to remember using the following argument. We can consider a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ as a collection of *bi-letters* $\substack{i\\j}$ as follows:

$$\pi = \left\{ \begin{matrix} 1 \\ \pi_1 \end{matrix}, \begin{matrix} 2 \\ \pi_2 \end{matrix}, \dots, \begin{matrix} n \\ \pi_n \end{matrix} \right\}.$$

Then $\pi_{\le k}$ (resp. $\pi_{>k}$, $\pi^{\le k}$ and $\pi^{>k}$) is the permutation obtained from π by taking bi-letters $\substack{i\\j}$ with $j \le k$ (resp. $j > k$, $i \le k$ and $i > k$) and by order-preserving relabeling. It is easy to see that $(\pi^{\le k})^{-1} = (\pi^{-1})_{\le k}$.

Jaggard [2] defined the j-set as follows. For a permutation π, the *j-set* $J(\pi)$ of π is defined to be the set of integers $j \ge 0$ such that $\pi^{\le j}$ is an involution, i.e,

$$J(\pi) = \{j : \pi^{\le j} = (\pi^{-1})_{\le j}\}.$$

Note that $\pi^{\le 0} = \emptyset$, the empty permutation, which we consider as an involution. Thus, we always have $0 \in J(\pi)$. Kim and Kim [3] found a criterion for a j-set.

Definition 2.1. Let σ and τ be permutations. The *j_2-set* $J(\sigma,\tau)$ is defined to be

$$J(\sigma,\tau) = \{j : \sigma^{\le j} = \tau_{\le j}\}.$$

Note that we always have $0 \in J(\sigma,\tau)$. Since $J(\pi) = J(\pi, \pi^{-1})$, every j-set is also a j_2-set. In Section 4, we will find a criterion for a j_2-set.

For $\pi \in \mathfrak{S}_n$, the *permutation matrix* $M(\pi)$ is the $n\times n$ matrix whose (i,j)-entry is 1 if $\pi_i = j$; and 0 otherwise. For example, $M(4132) = \begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix}$.

Let M be a 0-1 matrix such that each row and column contains at most one 1. Then, there exists a unique permutation π whose permutation matrix is obtained from M by removing the rows and columns consisting of zeroes. In that case, we write $\pi \sim M$. If $\pi \sim M$ and $\pi \sim N$, then we also write $M \sim N$. For example,

$$\begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&1 \end{pmatrix} \sim \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0 \end{pmatrix} \sim 213.$$

For an $n\times m$ matrix M, let $\mathrm{row}(M)$ (resp. $\mathrm{col}(M)$) denote the word $\mathbf{r} = r_1r_2\cdots r_n$ (resp. $\mathbf{c} = c_1c_2\cdots c_m$) of integers such that r_i (resp. c_i) is the sum of elements in the i-th row (resp. column) of M. If M is the second matrix above, then $\mathrm{row}(M) = 1011$ and $\mathrm{col}(M) = 11010$.

$$M(\pi) = \left(\begin{array}{cc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

FIGURE 1. The decomposition of $\pi = 7152436$ for $\phi_{2,3}$.

Let a, b, m, n, and ℓ be fixed integers such that $a + m = b + n = \ell$. Let $\pi \in \mathfrak{S}_\ell$. We divide the permutation matrix of π as follows:

$$M(\pi) = \begin{array}{c c} & \begin{array}{cc} a & \quad m \end{array} \\ \begin{array}{c} b \\ n \end{array} & \begin{pmatrix} M_{(1,1)} & M_{(1,2)} \\ M_{(2,1)} & M_{(2,2)} \end{pmatrix} \end{array},$$

where the numbers outside the matrix indicate the sizes of the block matrices.

Assume that $M_{(1,1)}$ contains j 1's. Then $M_{(1,2)}$, $M_{(2,1)}$ and $M_{(2,2)}$ contain $b-j$, $a-j$ and $n-a+j$ 1's respectively. Let $k = n-a+j$. Let $\pi_{(r,s)}$ be the permutation satisfying $\pi_{(r,s)} \sim M_{(r,s)}$ for $r = 1,2$ and $s = 1,2$. Let $\mathbf{c}_1 = \mathrm{col}(M_{(2,1)})$, $\mathbf{c}_2 = \mathrm{col}(M_{(2,2)})$, $\mathbf{r}_1 = \mathrm{col}(M_{(1,2)})$ and $\mathbf{r}_2 = \mathrm{col}(M_{(2,2)})$. Then we define the map

$$\phi_{a,b} : \mathfrak{S}_\ell \to \bigcup_{\substack{0 \le j \le a \\ k = n-a+j}} \mathfrak{S}_j \times \mathfrak{S}_{b-j} \times \mathfrak{S}_{a-j} \times \mathfrak{S}_k \times \binom{[a]}{a-j} \times \binom{[b]}{b-j} \times \binom{[m]}{k} \times \binom{[n]}{k}$$

by

$$\phi_{a,b}(\pi) = (\pi_{(1,1)}, \pi_{(1,2)}, \pi_{(2,1)}, \pi_{(2,2)}, \mathbf{c}_1, \mathbf{r}_1, \mathbf{c}_2, \mathbf{r}_2),$$

where $\binom{[n]}{k}$ denotes the set of words consisting of k 1's and $n-k$ 0's.

For example, if $\pi = 7152436$ then, see Figure 1, we have

$$\phi_{2,3}(\pi) = (1, 21, 1, 213, 01, 101, 11010, 0111).$$

It is easy to see that $\phi_{a,b}$ is a bijection.

Let $\sigma \in \mathfrak{S}_a$, $\tau \in \mathfrak{S}_b$ and $\mathbf{r} \in \binom{[a+b]}{b}$. The *shuffle* $\mathsf{sf}(\sigma, \tau; \mathbf{r})$ is the permutation in $\mathfrak{S}_{a+b}$ obtained from $\mathbf{r}$ by replacing the i-th 0 to σ_i and the j-th 1 to $a + \tau_j$ for $1 \le i \le a$ and $1 \le j \le b$. For example, $\mathsf{sf}(3142, 231; 0010110) = 3164752$. The following lemma is due to Garsia and Gessel [1].

Lemma 2.2. *Let a, b be integers and let $\sigma \in \mathfrak{S}_a$, $\tau \in \mathfrak{S}_b$. Then*

$$\sum_{\mathbf{r} \in \binom{[a+b]}{b}} q^{\mathrm{maj}(\mathsf{sf}(\sigma,\tau;\mathbf{r}))} = q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\tau)} \begin{bmatrix} a+b \\ b \end{bmatrix}_q.$$

Now we can prove Theorem 1.1 and 1.4. Recall that

$$t_n(q) = \sum_{\pi \in \mathfrak{I}_n} q^{\mathrm{maj}(\pi)}, \quad A_k(p,q) = \sum_{\pi \in \mathfrak{S}_k} p^{\mathrm{imaj}(\pi)} q^{\mathrm{maj}(\pi)},$$

and $\mathfrak{S}_n|_\sigma^\tau = \{\pi \in \mathfrak{S}_n : \pi_{\le a} = \sigma, \pi^{\le b} = \tau\}$.

Theorem 2.3. *Let a be an integers and let $\sigma \in \mathfrak{S}_a$. Then*

$$\sum_{\pi\in\mathfrak{I}_{n+a}} q^{\mathrm{maj}(\pi^{>a})} = \sum_{\substack{0\le j\le a\\ k=n-a+j}} t_j \binom{a}{j} [a-j]_q! \begin{bmatrix} n\\ k\end{bmatrix}_q t_k(q),$$

$$\sum_{\pi\in\mathfrak{I}_{n+a}(\sigma)} q^{\mathrm{maj}(\pi^{>a})} = \sum_{\substack{j\in J(\sigma)\\ k=n-a+j}} q^{\mathrm{maj}(\sigma^{>j})} \begin{bmatrix} n\\ k\end{bmatrix}_q t_k(q).$$

Proof. Similar to the proof of the following theorem. □

Theorem 2.4. *Let a, b, n, m and ℓ be integers with $a + m = b + n = \ell$. Let $\sigma \in \mathfrak{S}_a, \tau \in \mathfrak{S}_b$. Then*

$$\sum_{\pi\in\mathfrak{S}_\ell} p^{\mathrm{imaj}(\pi_{>a})} q^{\mathrm{maj}(\pi^{>b})} = \sum_{\substack{0\le j\le a\\ k=n-a+j}} j! \binom{a}{j}\binom{b}{j} [b-j]_p! \, [a-j]_q! \begin{bmatrix} m\\ k\end{bmatrix}_p \begin{bmatrix} n\\ k\end{bmatrix}_q A_k(p,q),$$

$$\sum_{\pi\in\mathfrak{S}_\ell|_\sigma^\tau} p^{\mathrm{imaj}(\pi_{>a})} q^{\mathrm{maj}(\pi^{>b})} = \sum_{\substack{j\in J(\sigma,\tau)\\ k=n-a+j}} p^{\mathrm{imaj}(\tau_{>j})} q^{\mathrm{maj}(\sigma^{>j})} \begin{bmatrix} m\\ k\end{bmatrix}_p \begin{bmatrix} n\\ k\end{bmatrix}_q A_k(p,q).$$

Proof. Let $0 \le j \le a$. Consider a permutation $\pi \in \mathfrak{S}_\ell$ such that $\pi_{(1,1)} \in \mathfrak{S}_j$, where

$$\phi_{a,b}(\pi) = (\pi_{(1,1)}, \pi_{(1,2)}, \pi_{(2,1)}, \pi_{(2,2)}, \mathbf{c}_1, \mathbf{r}_1, \mathbf{c}_2, \mathbf{r}_2).$$

Then $(\pi_{>a})^{-1} = \mathsf{sf}(\pi_{(1,2)}^{-1}, \pi_{(2,2)}^{-1}; \mathbf{c}_2)$ and $\pi^{>b} = \mathsf{sf}(\pi_{(2,1)}, \pi_{(2,2)}; \mathbf{r}_2)$. Thus

$$p^{\mathrm{imaj}(\pi_{>a})} q^{\mathrm{maj}(\pi^{>b})} = p^{\mathrm{maj}(\mathsf{sf}(\pi_{(1,2)}^{-1}, \pi_{(2,2)}^{-1}; \mathbf{c}_2))} q^{\mathrm{maj}(\mathsf{sf}(\pi_{(2,1)}, \pi_{(2,2)}; \mathbf{r}_2))}. \tag{6}$$

Let $k = n - a + j$. By Lemma 2.2, the sum of (6) over all $\pi_{(2,2)} \in \mathfrak{S}_k$, $\mathbf{c}_2 \in \binom{[m]}{k}$ and $\mathbf{r}_2 \in \binom{[n]}{k}$ equals

$$p^{\mathrm{imaj}(\pi_{(1,2)})} q^{\mathrm{maj}(\pi_{(2,1)})} \begin{bmatrix} m\\ k\end{bmatrix}_p \begin{bmatrix} n\\ k\end{bmatrix}_q A_k(p,q). \tag{7}$$

Summing (7) over all j, $\pi_{(1,1)} \in \mathfrak{S}_j$, $\pi_{(1,2)} \in \mathfrak{S}_{b-j}$, $\pi_{(2,1)} \in \mathfrak{S}_{a-j}$, $\mathbf{c}_1 \in \binom{[a]}{a-j}$ and $\mathbf{r}_1 \in \binom{[b]}{b-j}$, and using the well known result $\sum_{\pi\in\mathfrak{S}_n} q^{\mathrm{maj}(\pi)} = [n]_q!$, we get the first identity.

If $\pi \in \mathfrak{S}_\ell|_\sigma^\tau$, then $\mathsf{sf}(\pi_{(1,1)}, \pi_{(1,2)}; \mathbf{r}_1) = \tau$ and $\mathsf{sf}(\pi_{(1,1)}^{-1}, \pi_{(2,1)}^{-1}; \mathbf{c}_1) = \sigma^{-1}$, which implies $\pi_{(1,1)} = \sigma^{\le j} = \tau_{\le j}$, $\pi_{(1,2)} = \tau_{>j}$ and $\pi_{(2,1)} = \sigma^{>j}$. Thus we have $j \in J(\sigma,\tau)$, and j determines $\pi_{(1,1)}$, $\pi_{(1,2)}$, $\pi_{(2,1)}$, $\mathbf{c}_1$ and $\mathbf{r}_1$. Then we get the second identity by summing (7) over all $j \in J(\sigma,\tau)$. □

Recall that for a real number $r > 0$, we denote $\bar{r} = \min(r, r^{-1})$.

The proof of the following lemma is in Section 5.

Lemma 2.5. *Let $p, q > 0$. Then*

$$\lim_{n\to\infty} \frac{\frac{t_{n+1}(q)}{[n+1]_q!}}{\frac{t_n(q)}{[n]_q!}} = 1 - \bar{q}, \qquad \lim_{n\to\infty} \frac{\frac{A_{n+1}(p,q)}{[n+1]_p![n+1]_q!}}{\frac{A_n(p,q)}{[n]_p![n]_q!}} = (1-\bar{p})(1-\bar{q}).$$

Theorem 1.2 is a consequence of the following theorem.

Theorem 2.6. *Let a be an integer and let $\sigma \in \mathfrak{S}_a$. If $q > 0$, then*

$$\lim_{n\to\infty} \frac{\sum_{\pi\in\mathfrak{I}_n(\sigma)} q^{\mathrm{maj}(\pi^{>a})}}{\sum_{\pi\in\mathfrak{I}_n} q^{\mathrm{maj}(\pi^{>a})}} = \frac{\sum_{j\in J(\sigma)} q^{\mathrm{maj}(\sigma^{>j})} \begin{bmatrix} a\\ j\end{bmatrix}_q [j]_q! (1-\bar{q})^j}{\sum_{j=0}^a [a]_q! t_j \binom{a}{j} (1-\bar{q})^j}.$$

Proof. By Theorem 2.3, the left hand side is equal to

$$\lim_{n\to\infty} \frac{\sum_{\pi\in\mathfrak{I}_{n+a}(\sigma)} \frac{q^{\mathrm{maj}(\pi^{>a})}}{[n]_q!}}{\sum_{\pi\in\mathfrak{I}_{n+a}} \frac{q^{\mathrm{maj}(\pi^{>a})}}{[n]_q!}} = \lim_{n\to\infty} \frac{\sum_{j\in J(\sigma)} \frac{q^{\mathrm{maj}(\sigma^{>j})}}{[a-j]_q!} \frac{t_{n-a+j}(q)}{[n-a+j]_q!}}{\sum_{j=0}^{a} t_j \binom{a}{j} \frac{t_{n-a+j}(q)}{[n-a+j]_q!}}. \tag{8}$$

By Lemma 2.5, we have

$$\lim_{n\to\infty} \frac{\frac{t_{n-a+j}(q)}{[n-a+j]_q!}}{\frac{t_{n-a}(q)}{[n-a]_q!}} = (1-\bar{q})^j.$$

Then, we get the theorem by dividing the numerator and denominator of the right hand side of (8) by $\frac{t_{n-a}(q)}{[n-a]_q!}$, and by multiplying them by $[a]_q!$. □

Similarly, we can prove the following theorem, which implies Theorem 1.6.

Theorem 2.7. *Let a, b be integers and let $\sigma \in \mathfrak{S}_a$, $\tau \in \mathfrak{S}_b$. If $p, q > 0$, then*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\sum_{\pi\in\mathfrak{S}_n|_\sigma^\tau} p^{\mathrm{imaj}(\pi_{>a})} q^{\mathrm{maj}(\pi^{>b})}}{\sum_{\pi\in\mathfrak{S}_n} p^{\mathrm{imaj}(\pi_{>a})} q^{\mathrm{maj}(\pi^{>b})}} \\
&= \frac{\sum_{j\in J(\sigma,\tau)} p^{\mathrm{imaj}(\tau_{>j})} q^{\mathrm{maj}(\sigma^{>j})} \begin{bmatrix} b \\ j \end{bmatrix}_p \begin{bmatrix} a \\ j \end{bmatrix}_q [j]_p! \, [j]_q! (1-\bar{p})^j (1-\bar{q})^j}{\sum_{j=0}^{a} [b]_p! \, [a]_q! j! \binom{a}{j}\binom{b}{j} (1-\bar{p})^j (1-\bar{q})^j}.
\end{aligned}$$

3. Tableau containment

Jaggard [2] proved that for a SYT A of shape α,

$$\#\{\sigma : P(\sigma) = A, j \in J(\sigma)\} = \sum_{\mu\vdash j} f^{\alpha/\mu}. \tag{9}$$

The following is a generalization of (9).

Lemma 3.1. *Let A be SYT of shape α. Let j be a fixed nonnegative integer. Then*

$$\sum_{\sigma : \begin{cases} P(\sigma)=A \\ j\in J(\sigma) \end{cases}} q^{\mathrm{maj}(\sigma^{>j})} = \sum_{\mu\vdash j} f^{\alpha/\mu}(q).$$

Proof. Similar to the proof of the following lemma. □

We can also consider pairs of SYTs.

Lemma 3.2. *Let A and B be SYTs of shape α and β respectively. Let j be a fixed nonnegative integer. Then*

$$\sum_{(\sigma,\tau) : \begin{cases} P(\sigma)=A \\ Q(\tau)=B \\ j\in J(\sigma,\tau) \end{cases}} p^{\mathrm{imaj}(\tau_{>j})} q^{\mathrm{maj}(\sigma^{>j})} = \sum_{\mu\vdash j} f^{\beta/\mu}(p) f^{\alpha/\mu}(q).$$

Proof. Let $X = \{(\sigma,\tau) : P(\sigma) = A, Q(\tau) = B, j \in J(\sigma,\tau)\}$ and $Y = \{(U,V) : \mu \vdash j, sh(U) = \beta/\mu, sh(V) = \alpha/\mu\}$. It is sufficient to find a bijection $\psi : X \to Y$ such that if $\psi(\sigma,\tau) = (U,V)$, then $\mathrm{imaj}(\tau_{>j}) = \mathrm{maj}(U)$ and $\mathrm{maj}(\sigma^{>j}) = \mathrm{maj}(V)$.

We define $\psi(\sigma,\tau) = (U,V)$ by $U = P(\tau)_{>j}$ and $V = Q(\sigma)_{>j}$. Then we have $\mathrm{imaj}(\tau_{>j}) = \mathrm{maj}(U)$ and $\mathrm{maj}(\sigma^{>j}) = \mathrm{maj}(V)$.

To prove ψ is a bijection, it is sufficient to show that for $(U,V) \in Y$ there exists a unique pair $(\sigma,\tau) \in X$ satisfying $\psi(\sigma,\tau) = (U,V)$. Let $\alpha \vdash a$. Since $P(\sigma) = A$, $Q(\sigma)_{>j} = V$, by reversing the insertion algorithm $a-j$ times, we can find $\sigma_a, \sigma_{a-1}, \ldots, \sigma_{j+1}$ and $P(\sigma^{\le j})$. Since $P(\tau)_{\le j} = P(\tau_{\le j}) = P(\sigma^{\le j})$ and $P(\tau)_{>j} = U$, we can determine $P(\tau)$. Thus we get τ, from which we can determine $\sigma^{\le j} = \tau_{\le j}$. Thus we get σ, and there is a unique pair $(\sigma,\tau) \in X$ with $\psi(\sigma,\tau) = (U,V)$. □

Now we can prove the following theorem, which implies Theorem 1.3.

Theorem 3.3. *Let A be a fixed SYT of shape $\alpha \vdash a$. If $q > 0$, then*

$$\lim_{n\to\infty} \frac{\sum_{T\in\mathcal{T}_n(A)} q^{\mathrm{maj}(T_{>a})}}{\sum_{T\in\mathcal{T}_n} q^{\mathrm{maj}(T_{>a})}} = \frac{\sum_{j=0}^{a} \begin{bmatrix} a \\ j \end{bmatrix}_q [j]_q!(1-\bar{q})^j \sum_{\mu\vdash j} f^{\alpha/\mu}(q)}{\sum_{j=0}^{a} [a]_q! t_j \binom{a}{j} (1-\bar{q})^j}.$$

Proof. By the Robinson-Schensted correspondence (1), we have

$$\lim_{n\to\infty} \frac{\sum_{T\in\mathcal{T}_n(A)} q^{\mathrm{maj}(T_{>a})}}{\sum_{T\in\mathcal{T}_n} q^{\mathrm{maj}(T_{>a})}} = \lim_{n\to\infty} \frac{\sum_{\sigma:P(\sigma)=A} \sum_{\pi\in\mathfrak{I}_n(\sigma)} q^{\mathrm{maj}(\pi^{>a})}}{\sum_{\pi\in\mathfrak{I}_n} q^{\mathrm{maj}(\pi^{>a})}},$$

which is, by Theorem 2.6, equal to

$$\frac{\sum_{\sigma:P(\sigma)=A} \sum_{j\in J(\sigma)} q^{\mathrm{maj}(\sigma^{>j})} \begin{bmatrix} a \\ j \end{bmatrix}_q [j]_q!(1-\bar{q})^j}{\sum_{j=0}^{a} [a]_q! t_j \binom{a}{j} (1-\bar{q})^j}.$$

The numerator is equal to

$$\sum_{j=0}^{a} \begin{bmatrix} a \\ j \end{bmatrix}_q [j]_q!(1-\bar{q})^j \sum_{\sigma:\left\{\substack{P(\sigma)=A \\ j\in J(\sigma)}\right.} q^{\mathrm{maj}(\sigma^{>j})}.$$

By Lemma 3.1, we are done. $\square$

Similarly, we get the following, which implies Theorem 1.7.

Theorem 3.4. *Let A and B be SYTs of shape $\alpha \vdash a$ and $\beta \vdash b$ respectively. If $p, q > 0$, then*

$$\begin{aligned} &\lim_{n\to\infty} \frac{\sum_{(P,Q)\in\mathcal{T}_n(A,B)} p^{\mathrm{maj}(P_{>a})} q^{\mathrm{maj}(Q_{>b})}}{\sum_{(P,Q)\in\mathcal{T}_n(\emptyset,\emptyset)} p^{\mathrm{maj}(P_{>a})} q^{\mathrm{maj}(Q_{>b})}} \\ &= \frac{\sum_{j=0}^{a} \begin{bmatrix} b \\ j \end{bmatrix}_p \begin{bmatrix} a \\ j \end{bmatrix}_q [j]_p! \, [j]_q!(1-\bar{p})^j(1-\bar{q})^j \sum_{\mu\vdash j} f^{\beta/\mu}(p) f^{\alpha/\mu}(q)}{\sum_{j=0}^{a} [b]_p! \, [a]_q! j! \binom{a}{j}\binom{b}{j}(1-\bar{p})^j(1-\bar{q})^j}. \end{aligned}$$

Using (9), Jaggard [2] proved the following theorem of Sagan and Stanley [7]:

$$\sum_{\lambda/\alpha\vdash n} f^{\lambda/\alpha} = \sum_{k\geq 0} \binom{n}{k} t_k \sum_{\alpha/\mu\vdash n-k} f^{\alpha/\mu}. \tag{10}$$

We can prove a q-analog of (10).

Theorem 3.5. *Let α be a fixed partition. Then*

$$\sum_{\lambda/\alpha\vdash n} f^{\lambda/\alpha}(q) = \sum_{k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q t_k(q) \sum_{\alpha/\mu\vdash n-k} f^{\alpha/\mu}(q).$$

Proof. Let $\alpha \vdash a$. Let A be a SYT of shape α. Then the left hand side is equal to

$$\begin{aligned}
\sum_{T\in\mathcal{T}_{n+a}(A)} q^{\mathrm{maj}(T_{>a})} &= \sum_{\sigma:P(\sigma)=A}\sum_{\pi\in\mathfrak{I}_{n+a}(\sigma)} q^{\mathrm{maj}(\pi^{>a})} \\
&= \sum_{\sigma:P(\sigma)=A}\sum_{\substack{j\in J(\sigma)\\ k=n-a+j}} q^{\mathrm{maj}(\sigma^{>j})} \begin{bmatrix} n\\ k\end{bmatrix}_q t_k(q) \qquad \text{(by Theorem 2.3)} \\
&= \sum_{\substack{0\le j\le a\\ k=n-a+j}} \begin{bmatrix} n\\ k\end{bmatrix}_q t_k(q) \sum_{\sigma:\left\{\substack{P(\sigma)=A\\ j\in J(\sigma)}\right.} q^{\mathrm{maj}(\sigma^{>j})} \\
&= \sum_{\substack{0\le j\le a\\ k=n-a+j}} \begin{bmatrix} n\\ k\end{bmatrix}_q t_k(q) \sum_{\mu\vdash j} f^{\alpha/\mu}(q) \qquad \text{(by Lemma 3.1)} \\
&= \sum_{k\ge 0} \begin{bmatrix} n\\ k\end{bmatrix}_q t_k(q) \sum_{\alpha/\mu\vdash n-k} f^{\alpha/\mu}(q).
\end{aligned}$$

□

Sagan and Stanley [7] also proved the following:

$$\sum_{\substack{\lambda/\alpha\vdash m\\ \lambda/\beta\vdash n}} f^{\lambda/\alpha} f^{\lambda/\beta} = \sum_{k\ge 0}\binom{m}{k}\binom{n}{k} k! \sum_{\substack{\beta/\mu\vdash m-k\\ \alpha/\mu\vdash n-k}} f^{\beta/\mu} f^{\alpha/\mu}. \tag{11}$$

Using the same argument of Theorem 3.5, we can prove a q-analog of (11).

Theorem 3.6. *Let α and β be fixed partitions. Then*

$$\sum_{\substack{\lambda/\alpha\vdash m\\ \lambda/\beta\vdash n}} f^{\lambda/\alpha}(p) f^{\lambda/\beta}(q) = \sum_{k\ge 0} \begin{bmatrix} m\\ k\end{bmatrix}_p \begin{bmatrix} n\\ k\end{bmatrix}_q A_k(p,q) \sum_{\substack{\beta/\mu\vdash m-k\\ \alpha/\mu\vdash n-k}} f^{\beta/\mu}(p) f^{\alpha/\mu}(q).$$

We note that Theorem 3.5 and Theorem 3.6 can also be proved using the following identities of skew Schur functions:

$$\sum_{\lambda} s_{\lambda/\alpha}(\mathbf{x}) = \sum_{\mu} s_{\alpha/\mu}(\mathbf{x}) \prod_i (1-x_i)^{-1} \prod_{i<j} (1-x_i x_j)^{-1},$$

$$\sum_{\lambda} s_{\lambda/\alpha}(\mathbf{x}) s_{\lambda/\beta}(\mathbf{y}) = \prod_{i,j} (1-x_i y_j)^{-1} \sum_{\mu} s_{\beta/\mu}(\mathbf{x}) s_{\alpha/\mu}(\mathbf{y}).$$

4. Criterion for a j_2-set

Kim and Kim [3] found the following a criterion for a j-set.

Theorem 4.1. *Let J be a j-set with the largest element $m \ge 2$. Then, for $n > m$, $J \cup \{n\}$ is a j-set if and only if $n = m+1$ or $n - m \ge m - \max(J \cap [m-2])$.*

Since $J(\sigma) = J(\sigma, \sigma^{-1})$, a j-set is also a j_2-set. But the converse is not true. In this section we will find a criterion for a j_2-set. Our proof is similar to the proof of Theorem 4.1 in [3], but easier than that.

We start with a simple observation.

Proposition 4.2. *Let J be a j_2-set with the largest element n. Then there is a permutation in $\mathfrak{S}_n$ such that $J(\pi,\pi) = J$.*

Proof. Let (σ,τ) be a pair with $J = J(\sigma,\tau)$. Since $\sigma^{\le n} = \tau_{\le n}$, if we set $\pi = \sigma^{\le n}$, then we have $J(\pi,\pi) = J$. □

To prove the criterion theorem we need the following four lemmas. Recall that, for 0-1 matrices M and N, we write $M \sim N$ if the matrices obtained from M and N by removing rows and columns consisting of zeroes are the same.

For the rest of this section, we assume that n and k are positive integers.

Lemma 4.3. *Let m be an integer greater than 1. Let J be a j_2-set such that the three largest elements of J are $n-k$, n and $n+m$. Then $m \geq k$.*

Proof. Let π be a permutation in $\mathfrak{S}_{n+m}$ satisfying $J(\pi,\pi) = J$. Let A, B, C and D be the $n\times n$, $n\times m$, $m\times n$ and $m\times m$ matrices respectively such that $M(\pi) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Since $n \in J(\pi,\pi)$, we have $\begin{pmatrix} A & B \end{pmatrix} \sim \begin{pmatrix} A \\ C \end{pmatrix}$. Let $\sigma = \pi^{\leq n} = \pi_{\leq n}$. Let B have s nonzero entries. Then C also has s nonzero entries. If $s = 0$, then we get $n+1 \in J(\pi,\pi)$ because $\pi^{\leq n+1} \sim \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \sim \pi_{\leq n+1}$. But this is a contradiction to $m > 1$. Thus $s \geq 1$. Since $\sigma^{\leq n-s} \sim A \sim \sigma_{\leq n-s}$, we get $\sigma^{\leq n-s} = \sigma_{\leq n-s}$. Thus

$$\pi^{\leq n-s} = (\pi^{\leq n})^{\leq n-s} = \sigma^{\leq n-s} = \sigma_{\leq n-s} = (\pi_{\leq n})_{\leq n-s} = \pi_{\leq n-s}$$

and we get $n-s \in J(\pi,\pi)$. Since $n-k$ is the largest element in $J \cap [n-1]$, we get $n-s \leq n-k$. Since B has at most m nonzero entries, we get $k \leq s \leq m$. □

Lemma 4.4. *Let J be a j_2-set such that the two largest elements of J are $n-k$ and n. Then $J \cup \{n+k\}$ is a j_2-set.*

Proof. If $k=1$, then it is clear. Assume $k \geq 2$. Let σ be a permutation in $\mathfrak{S}_n$ with $J(\sigma,\sigma) = J$. Let $\pi \in \mathfrak{S}_{n+k}$ be the permutation satisfying

$$M(\pi) = \begin{pmatrix} A & C \\ B & \mathbf{0} \end{pmatrix}, \tag{12}$$

where A, B and C are the matrices of size $n\times n$, $k \times n$ and $n \times k$ respectively such that $M(\sigma) \sim \begin{pmatrix} A \\ B \end{pmatrix} \sim \begin{pmatrix} A & C \end{pmatrix}$. Since $J(\pi,\pi) \cap [n] = J$ and $n+k \in J(\pi,\pi)$, it is sufficient to show that $n+s \notin J(\pi,\pi)$ for all $1 \leq s < k$. Suppose $n+s \in J(\pi,\pi)$ for some $1 \leq s < k$. Then we have

$$\begin{pmatrix} A & C \\ B' & \mathbf{0} \end{pmatrix} \sim \begin{pmatrix} A & C' \\ B & \mathbf{0} \end{pmatrix}, \tag{13}$$

where B' (resp. C') is the matrix consisting of the first s rows of B (resp. columns of C). Removing the last $k-s$ nonzero rows and columns of the matrices in both sides of (13), we get $\sigma^{\leq n-k+s} = \sigma_{\leq n-k+s}$, i.e., $n-k+s \in J(\sigma,\sigma) = J$, which is a contradiction to the assumption that $n-k$ and n are the two largest element of J. □

Lemma 4.5. *Let J be a j_2-set such that the two largest elements of J are $n-k$ and n. If $k \geq 2$, then $(J \setminus \{n\}) \cup \{n+1\}$ is a j_2-set.*

Proof. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation in $\mathfrak{S}_n$ with $J(\sigma,\sigma) = J$. Since $n-1 \notin J(\sigma,\sigma)$, we have $\sigma_n \neq n$. Let $\pi \in \mathfrak{S}_{n+1}$ be the permutation such that

$$\pi_i = \begin{cases} \sigma_i & \text{if } i < n \text{ and } \sigma_i < n, \\ n+1 & \text{if } i < n \text{ and } \sigma_i = n, \\ n & \text{if } i = n, \\ \sigma_n & \text{if } i = n+1. \end{cases}$$

Then $M(\sigma)$ and $M(\pi)$ are decomposed as follows:

$$M(\sigma) = \begin{array}{|c|c|} \hline & \\ A & C \\ & \\ \hline B & 0 \\ \hline \end{array}\,, \quad M(\pi) = \begin{array}{|c|c|c|} \hline & 0 & \\ A & \vdots & C \\ & 0 & \\ \hline 0 \ \cdots \ 0 & 1 & 0 \\ \hline B & 0 & 0 \\ \hline \end{array}\,,$$

where A, B and C are $(n-1)\times(n-1)$, $1\times(n-1)$ and $(n-1)\times 1$ matrices respectively. It is not difficult to see that $J(\pi,\pi) = (J \setminus \{n\}) \cup \{n+1\}$. $\square$

Lemma 4.6. *Let J be a j_2-set such that the two largest elements of J are $n-1$ and n. Then $J \cup \{n+k\}$ is a j_2-set for any positive integer k.*

Proof. It is clear if $k=1$. Assume $k \geq 2$. Let $\sigma \in \mathfrak{S}_{n-1}$ with $J(\sigma,\sigma) = J \setminus \{n\}$. Let $\pi \in \mathfrak{S}_{n+k}$ such that $M(\pi) = \begin{pmatrix} M(\sigma) & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix}$, where $A = \begin{pmatrix} \mathbf{0} & I_{k-1} \\ 1 & \mathbf{0} \end{pmatrix}$ and I_{k-1} is the $(k-1)\times(k-1)$ identity matrix. Then $J(\pi,\pi) = J \cup \{n+k\}$. $\square$

Summarizing the above four lemmas, we obtain the following criterion theorem.

Theorem 4.7. *Let J be a j_2-set such that the two largest elements of J are $n-k$ and n. Then $J \cup \{n+m\}$ is a j_2-set if and only if $m=1$ or $m \geq k$.*

It is easy to see that if J is a j_2-set, then $J \cap [k]$ is also a j_2-set for any integer k. Using Theorem 4.7, we can find a simple method to check whether a given set is a j_2-set or not.

Let $S = \{s_0, s_1, s_2, \ldots, s_n\}$ be a set of integers such that $s_0 < s_1 < \cdots < s_n$. We define $\Delta(S)$ as follows. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be the sequence such that $a_i = s_{n-i+1} - s_{n-i}$. Let $\mathbf{i} = \{i_1, i_2, \ldots, i_k\}$ be the set of integers such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $a_j = 1$ if and only if $j \in \mathbf{i}$ and let $i_0 = 0$. Then $\Delta(S)$ is the sequence $(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k)$, where $\mathbf{s}_j = (a_{i_{j-1}+1}, a_{i_{j-1}+2}, \ldots, a_{i_j})$.

Example 4.8. Let $S = \{0,1,3,6,7,8,12,13,14,15,17\}$. Then $\mathbf{a} = \{2,1,1,1,4,1,1,3,2,1\}$. Thus $\Delta(S) = ((2,1),(1),(1),(4,1),(1),(3,2,1))$.

Corollary 4.9. *Let $S = \{s_0, s_1, s_2, \ldots, s_n\}$ be a set of integers such that $s_0 < s_1 < \cdots < s_n$ and $\Delta(S) = (\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k)$. Then S is a j_2-set if and only if $s_0 = 0$, $s_1 = 1$ and $\mathbf{s}_i$ is a partition with exactly one part equal to 1 for all $i \in [k]$.*

Proof. It is a straightforward verification using Theorem 4.7. $\square$

By Corollary 4.9, the set S in Example 4.8 is a j_2-set. Using Corollary 4.9, we can easily get the generating function for the number of j_2-sets.

Corollary 4.10. *Let $j_2(n)$ be the number of j_2-sets with the largest element n. Then*

$$\sum_{n\geq 0} j_2(n)x^n = \frac{x\prod_{i\geq 2}\frac{1}{1-x^i}}{1 - x\prod_{i\geq 2}\frac{1}{1-x^i}} = \frac{x}{\prod_{i\geq 2}(1-x^i) - x},$$

$$\{j_2(n)\}_{n\geq 1} = \{1,1,2,4,8,15,29,55,105,200,381,725,1381,2629,5005,\ldots\}.$$

5. Proof of Lemma 2.5

Lemma 5.1. *We have*

$$\frac{t_n(q^{-1})}{[n]_{q^{-1}}!} = \frac{t_n(q)}{[n]_q!},$$

$$\frac{A_n(p^{-1},q)}{[n]_{p^{-1}}!\,[n]_q!} = \frac{A_n(p,q^{-1})}{[n]_p!\,[n]_{q^{-1}}!} = \frac{A_n(p^{-1},q^{-1})}{[n]_{p^{-1}}!\,[n]_{q^{-1}}!} = \frac{A_n(p,q)}{[n]_p!\,[n]_q!}.$$

Proof. Let T be a SYT of size n. Then $\mathrm{maj}(T)+\mathrm{maj}(T')=\binom{n}{2}$, where T' denotes the transpose of T. Thus

$$t_n(q^{-1})=\sum_{T\in\mathcal{T}_n} q^{-\mathrm{maj}(T)} = q^{-\binom{n}{2}}\sum_{T\in\mathcal{T}_n} q^{\mathrm{maj}(T')} = q^{-\binom{n}{2}} t_n(q).$$

Since $q^{-\binom{n}{2}}/\left[n\right]_{q^{-1}}! = 1/\left[n\right]_q!$, we get $\frac{t_n(q^{-1})}{[n]_{q^{-1}}!}=\frac{t_n(q)}{[n]_q!}$. The rest identities can be proved similarly. □

Lemma 5.2. *If $0<q<1$, then*

$$\log\left(\prod_{i\ge1}(1-q^i)^{-i}\right) < \left(1+\frac{q}{(1-q)^2}\right)\left(1+\log\frac{1}{1-q}\right).$$

Proof. The left hand side is equal to

$$\begin{aligned}
&\sum_{i\ge1} i\log\left(\frac{1}{1-q^i}\right)=\sum_{i\ge1} i\sum_{j\ge1}\frac{q^{ij}}{j}\\
&=\sum_{i,j\ge2}\frac{iq^{ij}}{j}+\sum_{i\ge1} iq^i+\sum_{j\ge1}\frac{q^j}{j}-1\\
&<\sum_{i,j\ge2}\frac{iq^{i+j}}{j}+\sum_{i\ge1} iq^i+\sum_{j\ge1}\frac{q^j}{j}\\
&<\left(1+\sum_{i\ge1} iq^i\right)\left(1+\sum_{j\ge1}\frac{q^j}{j}\right)=\left(1+\frac{q}{(1-q)^2}\right)\left(1+\log\frac{1}{1-q}\right).
\end{aligned}$$

□

Let $\mathbf{x}=(x_1,x_2,\dots)$ and $\mathbf{y}=(y_1,y_2,\dots)$ be two infinite sequences of independent variables. Let $s_\lambda(\mathbf{x})$ denote the Schur function in the variables $\mathbf{x}$. The following formulas are well known, see [4, 8].

$$\sum_{n\ge0}\sum_{\lambda\vdash n} s_\lambda(\mathbf{x})z^n=\prod_{i\ge1}(1-x_iz)^{-1}\prod_{1\le i<j}(1-x_ix_jz^2)^{-1}, \tag{14}$$

$$\sum_{n\ge0}\sum_{\lambda\vdash n} s_\lambda(\mathbf{x})s_\lambda(\mathbf{y})z^n=\prod_{i,j\ge1}(1-x_iy_jz)^{-1}. \tag{15}$$

It is also known, see [8, Proposition 7.19.11], that if $x_i=q^{i-1}$, then

$$s_\lambda(1,q,q^2,\dots)=\frac{f^\lambda(q)}{(1-q)^n\left[n\right]_q!}.$$

Thus we get the following.

$$\sum_{\lambda\vdash n} s_\lambda(1,q,q^2,\dots)=\frac{t_n(q)}{(1-q)^n\left[n\right]_q!} \tag{16}$$

$$\sum_{\lambda\vdash n} s_\lambda(1,p,p^2,\dots)s_\lambda(1,q,q^2,\dots)=\frac{A_n(p,q)}{(1-p)^n(1-q)^n\left[n\right]_p!\left[n\right]_q!} \tag{17}$$

Lemma 5.3. *Let $0<p,q<1$. Then*

$$\lim_{n\to\infty}\sum_{\lambda\vdash n} s_\lambda(1,q,q^2,\dots)=\prod_{i\ge1}(1-q^i)^{-1}\prod_{0\le i<j}(1-q^{i+j})^{-1},$$

$$\lim_{n\to\infty}\sum_{\lambda\vdash n} s_\lambda(1,p,p^2,\dots)s_\lambda(1,q,q^2,\dots)=\prod_{\substack{i,j\ge0\\ i+j>0}}(1-p^iq^j)^{-1}.$$

Proof. Let $\xi_n(q) = \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots)$. By (14), we have

$$\sum_{n\geq 0} \xi_n(q) z^n = \prod_{i\geq 0}(1-q^i z)^{-1} \prod_{0\leq i<j}(1-q^{i+j}z^2)^{-1},$$

equivalently,

$$\sum_{n\geq 0}(\xi_n(q)-\xi_{n-1}(q))z^n = \prod_{i\geq 1}(1-q^i z)^{-1} \prod_{0\leq i<j}(1-q^{i+j}z^2)^{-1},$$

where $\xi_{-1}(q) = 0$.

Then

$$\lim_{N\to\infty} \xi_N(q) = \lim_{N\to\infty} \sum_{n=0}^{N}(\xi_n(q)-\xi_{n-1}(q)) = \prod_{i\geq 1}(1-q^i)^{-1} \prod_{0\leq i<j}(1-q^{i+j})^{-1}$$

converges, because

$$\prod_{i\geq 1}(1-q^i)^{-1} < \prod_{0\leq i<j}(1-q^{i+j})^{-1} = \prod_{i\geq 1}(1-q^i)^{-\lceil \frac{i}{2} \rceil} < \prod_{i\geq 1}(1-q^i)^{-i},$$

where $\prod_{i\geq 1}(1-q^i)^{-i}$ converges by Lemma 5.2. Thus we get the first limit. Similarly, we can prove the second limit. $\square$

Proof of Lemma 2.5. We will only prove the first limit. The second can be proved similarly. Using the well known asymptotic behavior of $t_n \sim \frac{1}{\sqrt{2}} n^{n/2} \exp\left(-\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right)$, we can easily see that it holds for $q = 1$. Assume $q \neq 1$. By Lemma 5.1, it is sufficient to show that for $0 < q < 1$,

$$\lim_{n\to\infty} \frac{\frac{t_{n+1}(q)}{[n+1]_q!}}{\frac{t_n(q)}{[n]_q!}} = 1 - q.$$

Since $\frac{t_n(q)}{[n]_q!} = (1-q)^n \sum_{\lambda\vdash n} s_\lambda(1, q, q^2, \ldots)$, we are done by Lemma 5.3. $\square$

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